The classical limit of the quantum Zeno effect

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Abstract
The evolution of a quantum system subjected to infinitely many measurements in a finite time interval is confined in a proper subspace of the Hilbert space. This phenomenon is called 'quantum Zeno effect': a particle under intensive observation which does not evolve. This effect is at variance with the classical evolution, which obviously is not affected by any observations. By a semiclassical analysis, we will show that the quantum Zeno effect vanishes at all orders, when the Planck constant tends to zero, and thus it is a purely quantum phenomenon without classical analog, at the same level of tunneling.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In this communication, we study the classical limit of the quantum Zeno effect in its simplest formulation, namely a free particle subjected to position measurements. The presence of any smooth and bounded potential does not affect our results.

Therefore, let us consider a free quantum particle in $\mathbb{R}^n$. Its states are described by vectors in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$ and the Schrödinger operator is $H = -\hbar^2 \Delta /2m$ with the domain being the Sobolev space $D(H) = H^2(\mathbb{R}^n)$. Let $P = \chi_\Omega$ be the orthogonal projection onto a compact set $\Omega \subset \mathbb{R}^n$ with regular boundary. Here χ_Ω denotes the characteristic function of the set Ω ($\chi_\Omega(x)$ equals 1 for $x \in \Omega$ and 0 otherwise). P is the observable associated to a measurement that ascertains whether or not the particle is in the spatial region Ω. If one performs N measurements on the particle at regular time intervals of length t/N, at the end of this procedure the state of the system is, up to a normalization,

$$\psi_N(t) = (P\,U(t/N)\,P)^N \psi, \tag{1}$$

where ψ is the initial state of the particle and $U(t) = \mathrm{e}^{-\mathrm{i}tH/\hbar}$ is the evolution group generated by H. Let

$$V_N(t) = (P\,\mathrm{e}^{-\mathrm{i}tH/\hbar N}\,P)^N. \tag{2}$$

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We are interested in the limit $N \rightarrow +\infty$ of the product formula $V_N(t)$. In [4] it has been proved that

Theorem 1. *There exists a set $M \subset \mathbb{R}$ of Lebesgue measure zero and a strictly increasing sequence $\{N\}$ of positive integers along which we have*

$$\lim_{N \rightarrow +\infty} V_N(t)\psi = \mathrm{e}^{-\mathrm{i}t H_\Omega/\hbar} P\psi,$$

for all $\psi \in \mathcal{H}$ and for all $t \in \mathbb{R} \setminus M$, where $H_\Omega = -\hbar^2 \Delta_\Omega/2m$, and Δ_Ω is the Laplace operator with Dirichlet boundary condition on $\partial\Omega$, that is $D(H_\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$.

The limit in theorem 1 implies that, if it is possible to perform infinitely many position measurements in the finite time interval $[0, t]$, the probability of finding the particle in the region Ω in *each* of these measurements reads

$$p_N(t) = \langle \psi_N(t), \psi_N(t) \rangle = \|V_N(t)\psi\|^2 \rightarrow \|P\psi\|^2 = 1, \tag{3}$$

for $N \rightarrow +\infty$. This peculiar quantum behavior was named *quantum Zeno effect* by Misra and Sudarshan [10]. Since then, the quantum Zeno effect has received constant attention by physicists and mathematicians. For an up-to-date review of the main mathematical and physical aspects, see [5] and references therein.

The effect has been observed experimentally in a variety of systems, on experiments involving photons [8], nuclear spins [12], ions [2], optical pumping [11], photons in a cavity [1], ultracold atoms [6] and Bose–Einstein condensates [15]. Moreover, these ideas might lead to remarkable applications, e.g. in quantum computation and in the control of decoherence.

Of course, the behavior in (3) is at complete variance with that of a classical particle. Indeed, a free particle with a nonzero initial momentum will eventually escape from the region Ω and obviously its motion is not modified by any observations. More precisely, a particle with initial momentum $\xi \neq 0$ after a time

$$T_\xi = m\delta(\Omega)/|\xi| \tag{4}$$

will be surely found outside Ω, independently of its initial position $x \in \Omega$, where

$$\delta(\Omega) = \sup_{x,y \in \Omega} |x - y| \tag{5}$$

is the diameter of Ω.

In this paper, we will prove that the quantum Zeno effect is a purely quantum phenomenon, at the same level of tunneling; namely, it cannot be observed at any finite order in $\hbar$, in the limit $\hbar \rightarrow 0$. Note that, in order to compare classical and quantum dynamics one has to describe them in the same space. In fact, by the Wigner–Moyal formalism, one can give a description of quantum mechanics in classical phase space, which is completely equivalent to the usual description in Hilbert space. Functions $\tau(x, \xi)$ on the phase space $\mathbb{R}^{2n}$ (classical observables) are mapped into operators $T = \mathrm{Op}^W(\tau)$ on the Hilbert space $L^2(\mathbb{R}^n)$ (quantum observables) via the Weyl quantization map. In particular, the noncommutative product of two quantum operators $T_1 T_2$ corresponds to the twisted convolution product (definition recalled below) $\tau_1 \sharp \tau_2$ of the classical observables τ_1 and τ_2, while the commutator $[T_1, T_2]$ corresponds to the Moyal bracket $\{\tau_1, \tau_2\}_M$. The main point here is that both $\tau_1 \sharp \tau_2$ and $\{\tau_1, \tau_2\}_M$ depend on the Planck constant $\hbar$. When $\hbar \rightarrow 0$ they reduce to commutative multiplication and Poisson bracket, respectively, thus restoring classical mechanics. Semiclassical analysis deals with all quantum corrections to classical mechanics at each order in $\hbar$, which are encoded in the asymptotic power series in $\hbar$ of $\tau_1 \sharp \tau_2$ and $\{\tau_1, \tau_2\}_M$.

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In the following, we will analyze (a suitable regularization of) the product formula in (2) with the above-mentioned tools. Let

$$\tilde{V}_N(t) = P_N(t)\, P_N\left(\frac{N-1}{N}t\right)\cdots P_N\left(\frac{2}{N}t\right) P_N\left(\frac{1}{N}t\right) P_N(0), \tag{6}$$

where P_N is the multiplication operator by a suitable C^∞ mollification of the characteristic function χ_Ω (see section 3) and $P_N(s)$, $s \in \mathbb{R}$, is the evolution of P_N in the Heisenberg picture. Our goal is to prove the following.

Theorem 2. *Let $M \subset \mathbb{R}$, $\{N\}$ and H_Ω be as in the theorem 1.*

(i)
$$\lim_{N \to +\infty} \tilde{V}_N(t)\psi = e^{itH/\hbar}\, e^{-itH_\Omega/\hbar} P\psi,$$

for all $\psi \in \mathcal{H}$ and for all $t \in \mathbb{R} \setminus M$.

(ii) $\tilde{V}_N(t)$ has a semiclassical symbol Θ_N and

$$\Theta_N(x, \xi; t) \sim \sum_{j=0}^{+\infty} \hbar^j \Theta_{j,N}(x, \xi; t),$$

for $\hbar \to 0$ and for all $x, \xi \in \mathbb{R}^n$.

(iii) For every $\xi \in \mathbb{R}^n$, $\xi \neq 0$, if $t > T_\xi := m\delta(\Omega)/|\xi|$, one has

$$\lim_{N \to +\infty} \Theta_{j,N}(x, \xi; t) = 0,$$

uniformly in $x \in \mathbb{R}^n$ and $j \in \mathbb{N}$.

Statement (i) of theorem 2 allows one to replace the product formula (2) with its regularized version (6), which is more suitable to a semiclassical analysis. Note, indeed, that for any $\psi \in \mathcal{H}$, also $\tilde{\psi}_N(t) = \tilde{V}_N(t)\psi$ satisfies equation (3) and thus is related to the probability of finding the particle in Ω in all N measurements.

Statement (ii) says that the quantum product formula $\tilde{V}_N(t)$ has a classical counterpart $\Theta_N(x, \xi; t)$ that admits an asymptotic expansion in $\hbar$, and, finally, statement (iii) asserts that each term $\Theta_{j,N}$ of the expansion identically vanishes for $t > T_\xi$ in the limit $N \to \infty$.

This last statement is the main result of this paper. Its physical meaning is the following: we consider the asymptotic expansion of the product formula for $\hbar \to 0$. About the zeroth order, classical, term we have already discussed: given an initial momentum $\xi \neq 0$, at times $t > T_\xi$ we get $\lim_{N \to \infty} \Theta_{0,N}(x, \xi; t) = 0$, uniformly in $x \in \Omega$, that is the classical particle, initially in Ω, has eventually escaped from that region. Statement (iii) asserts that the *same* feature is shared by *all* quantum corrections, independently of the order in $\hbar$.

2. Weyl's quantization and Egorov's theorem

In this section, mainly intended as a setup of the notation, we will briefly recall the tools needed in the following. For all details and proofs we refer to [3, 7, 13, 14, 16].

Let us start with Weyl's quantization. Let τ be a function in the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$. We can define the following operator on $L^2(\mathbb{R}^n)$:

$$\mathrm{Op}^W(\tau) = \int_{\mathbb{R}^{2n}} e^{i(\xi \cdot X + y \cdot p)} \hat{\tau}(\xi, y)\, \frac{d\xi\, dy}{(2\pi)^n}, \tag{7}$$

where X is the position operator $X\psi(x) = x\psi(x)$, $p = \hbar D_x/i$ the momentum operator, with D_x the n-dimensional gradient, and the Fourier transform is defined by

$$\hat{\tau}(\xi, y) = \int_{\mathbb{R}^{2n}} \tau(x, \eta)\, e^{-i(\xi \cdot x + \eta \cdot y)}\, \frac{dx\, d\eta}{(2\pi)^n}. \tag{8}$$

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It is easy to check that if τ is real then $\mathrm{Op}^W(\tau)$ is a bounded self-adjoint operator. The operator $\mathrm{Op}^W(\tau)$ is called the (Weyl) quantization of the symbol τ. Physically, it is interpreted as the quantum observable corresponding to the classical observable τ.

One can prove that, for any $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$(\mathrm{Op}^W(\tau)\psi)(x) = \int_{\mathbb{R}^{2n}} \tau\left(\frac{x+y}{2}, \xi\right) \mathrm{e}^{-\mathrm{i}\xi\cdot(y-x)/\hbar} \psi(y) \frac{\mathrm{d}\xi\,\mathrm{d}y}{(2\pi\hbar)^n}. \tag{9}$$

Equation (9) allows one to extend the quantization map to tempered distributions τ. We also recall the definition of the twisted convolution product between two symbols τ_1 and τ_2:

$$\tau_1 \sharp \tau_2(x,\xi) = \int_{\mathbb{R}^{4n}} \tau_1(x_1,\xi_1)\tau_2(x_2,\xi_2)\, \mathrm{e}^{\frac{2\mathrm{i}}{\hbar}[(x-x_1)\cdot(\xi-\xi_2)-(x-x_2)\cdot(\xi-\xi_1)]} \frac{\mathrm{d}x_1\,\mathrm{d}\xi_1\,\mathrm{d}x_2\,\mathrm{d}\xi_2}{(\pi\hbar)^{2n}}. \tag{10}$$

The twisted product is the image on the space of symbols of the noncommutative operator product, namely

$$\mathrm{Op}^W(\tau_1 \sharp \tau_2) = \mathrm{Op}^W(\tau_1)\,\mathrm{Op}^W(\tau_2). \tag{11}$$

The last ingredient we need in our analysis is Egorov's theorem that tell us how the time evolution and the Weyl quantization are related. We will focus our attention to the case we are interested in, i.e. the free Hamiltonian. In this case the Schrödinger operator $H = \mathrm{Op}^W(\mathcal{H})$ is the Weyl quantization of the Hamiltonian $\mathcal{H}(x,\xi) = \xi^2/2m$. The time evolution of a classical bounded observable is $\tau_t := \tau \circ \phi_t^{\mathcal{H}}$, where

$$\phi_t^{\mathcal{H}} : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \qquad \phi_t^{\mathcal{H}}(x,\xi) = \left(x + \frac{\xi t}{m}, \xi\right) \tag{12}$$

is the Hamiltonian flow. On the other hand, the quantum time evolution of a bounded observable $T = \mathrm{Op}^W(\tau)$ is

$$T(t) = \mathrm{e}^{\mathrm{i}tH/\hbar} T\, \mathrm{e}^{-\mathrm{i}tH/\hbar}, \tag{13}$$

which is a solution of the equation

$$\dot{T}(t) = \frac{\mathrm{i}}{\hbar}[H, T(t)]. \tag{14}$$

Let $\tau_t^{\hbar}(x,\xi)$ be the symbol of $T(t)$, namely $T(t) = \mathrm{Op}^W(\tau_t^{\hbar})$. Equation (14) is mirrored into the following equation for the symbol $\tau_t^{\hbar}$ on the phase space

$$\dot{\tau}_t^{\hbar} = \left\{\mathcal{H}, \tau_t^{\hbar}\right\}_M \tag{15}$$

with the initial condition $\tau_0^{\hbar} = \tau$, where

$$\{f, g\}_M = f \sharp g - g \sharp f, \tag{16}$$

is the Moyal bracket. Solving equation (15) one finds that, since $\mathcal{H}$ is quadratic in (x,ξ),

$$\tau_t^{\hbar} = \tau_t = \tau \circ \phi_t^{\mathcal{H}}, \tag{17}$$

namely

$$T(t) = \mathrm{e}^{\mathrm{i}tH/\hbar} T\, \mathrm{e}^{-\mathrm{i}tH/\hbar} = \mathrm{Op}^W(\tau_t^{\hbar}) = \mathrm{Op}^W(\tau \circ \phi_t^{\mathcal{H}}). \tag{18}$$

Thus, in this case, time evolution and quantization commute. For general nonquadratic Hamiltonian, the semiclassical Egorov theorem (see [13]) states that (17) holds only at order 0 in $\hbar$.
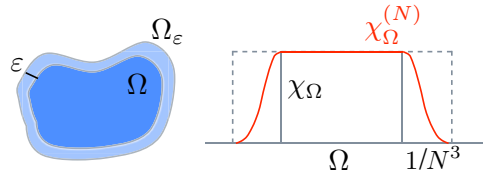
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Figure 1. ε-neighborhood of the compact set Ω and mollified characteristic function $\chi_\Omega^{(N)}$, with $\varepsilon_N = 1/N^3$.

3. A modified product formula

The projection P can be considered as a pseudodifferential operator whose symbol is the characteristic function $\varsigma(x, \xi) = \chi_\Omega(x)$ of the set $\Omega \times \mathbb{R}^n$ in the phase space. However, in order to have a sufficiently smooth symbol, instead of the projection P, we consider an operator $0 \leqslant P_N \leqslant 1$ as the Weyl quantization of a symbol which is a C^∞ mollification of the characteristic function χ_Ω. Namely, given an ε-neighbourhood of the domain Ω,

$$\Omega_\varepsilon = \{x \in \mathbb{R}^n \mid d(x, \Omega) < \varepsilon\}, \tag{19}$$

with $d(x, \Omega) = \inf_{y \in \Omega} |x - y|$ (see figure 1), we take

$$P_N = \mathrm{Op}^W(\vartheta^{(N)}), \qquad \vartheta^{(N)}(x, \xi) = \chi_\Omega^{(N)}(x), \tag{20}$$

where

$$\chi_\Omega \leqslant \chi_\Omega^{(N)} \leqslant \chi_{\Omega_{1/N^3}}, \qquad \chi_\Omega^{(N)} \in C^\infty(\mathbb{R}^n) \tag{21}$$

is a smoothed approximation of the characteristic function χ_Ω supported in Ω_{ε_N}, with $\varepsilon_N = 1/N^3$. See figure 1.

Observe now that, since $N(P_N - P)u \to 0$ when $N \to \infty$, one has that

$$N(P_N \, \mathrm{e}^{-itH/\hbar N} P_N - P \, \mathrm{e}^{-itH/\hbar N} P)u \to 0, \tag{22}$$

for any u such that $u, Pu \in H^2(\mathbb{R}^n)$, which is a dense subset of $L^2(\mathbb{R}^n)$. Therefore, the limit generators of the two discrete semigroups coincide. By theorem 1 it follows that

Lemma 1. *Let M, $\{N\}$ and H_Ω be as in theorem 1. One has*

$$\lim_{N \to +\infty} (P_N \, \mathrm{e}^{-itH/\hbar N} P_N)^N \psi = \mathrm{e}^{-itH_\Omega/\hbar} P\psi,$$

for all $\psi \in \mathcal{H}$ and for all $t \in \mathbb{R} \setminus M$.

Therefore, in our analysis of the quantum Zeno effect, we can use the sequence $\{P_N\}$ in place of the projection P. Note that, while the projection P is associated with a yes/no spatial measurement which ascertains whether or not the particle is in the region Ω, its smoothed version P_N corresponds to a fuzzy spatial measurement which is not sharp at the boundary of the region. Thus, the physical meaning of the above statement is that the interference effects arising from a small smoothing of the projection do not affect the overall phenomenon.

First let us rewrite $V_N(t)$ in a more convenient way. By using the evolution of P in the Heisenberg picture,

$$P(s) = \mathrm{e}^{isH/\hbar} P \, \mathrm{e}^{-isH/\hbar}, \tag{23}$$

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we obtain

$$V_N(t) = (P\,e^{-itH/N\hbar}\,P)^N$$
$$= e^{-itH/\hbar}\,P(t)\,P\left(\frac{N-1}{N}t\right)\cdots P\left(\frac{2}{N}t\right) P\left(\frac{1}{N}t\right) P(0). \qquad (24)$$

Now let us substitute in the above equation the projection P with the positive operator P_N given by equation (20) and neglect the final (trivial) unitary evolution in (24). We end up with the following product formula:

$$\tilde{V}_N(t) = P_N(t)\,P_N\left(\frac{N-1}{N}t\right)\cdots P_N\left(\frac{2}{N}t\right) P_N\left(\frac{1}{N}t\right) P_N(0). \qquad (25)$$

This is the main object of our investigation.

Note now that from theorem 1 and lemma 1 we immediately get the following.

Corollary 1. *One gets*

$$\lim_{N\to+\infty} \tilde{V}_N(t)\psi = e^{itH/\hbar}\,e^{-itH_\Omega/\hbar}\,P\psi,$$

for all $\psi \in \mathcal{H}$ and for all $t \in \mathbb{R} \setminus M$.

This is a reformulation of the quantum Zeno effect: the strong limit of the product formula (25) exists and yields a nontrivial evolution. In particular, note that, for any $\psi \in \mathcal{H}$, also $\tilde{\psi}_N(t) = \tilde{V}_N(t)\psi$ satisfies equation (3). Observe that corollary 1 is statement (*i*) of theorem 2.

4. Semiclassical analysis of the quantum Zeno effect

Now we have all the ingredients to prove the last statements of theorem 2. Let us focus on the classical limit of the product formula (25). First we can construct $\vartheta^{(N)}(x,\xi;t) = \big(\vartheta^{(N)} \circ \phi_t^{\mathcal{H}}\big)(x,\xi)$, which, since the Hamiltonian is quadratic, coincides with the symbol of the Heisenberg evolution of P_N. Define for all $k = 0, \ldots, N$,

$$\vartheta_k(x,\xi) := \vartheta^{(N)}\left(x,\xi; \frac{kt}{N}\right), \qquad (26)$$

so that the symbol of the operator (25), $\tilde{V}_N(t) = \mathrm{Op}^W(\Theta_N)$, is given by

$$\Theta_N = \vartheta_N \sharp \vartheta_{N-1} \sharp \cdots \sharp \vartheta_1 \sharp \vartheta_0. \qquad (27)$$

From equation (10), it is not difficult to show that [13, 16]

$$\phi_1 \sharp \phi_2 \sim \sum_{j=0}^{+\infty} \left(\frac{i\hbar}{2}\right)^j \frac{1}{j!}(D_{x,\phi_1}\cdot D_{\xi,\phi_2} - D_{x,\phi_2}\cdot D_{\xi,\phi_1})^j \phi_1 \phi_2$$
$$= \sum_{j=0}^{+\infty} \left(\frac{i\hbar}{2}\right)^j \frac{1}{j!}\phi_1 \sharp_j \phi_2, \qquad (28)$$

where

$$\phi_1 \sharp_j \phi_2 := (D_{x,\phi_1}\cdot D_{\xi,\phi_2} - D_{x,\phi_2}\cdot D_{\xi,\phi_1})^j \phi_1 \phi_2. \qquad (29)$$

Here, the subscripts ϕ_1 and ϕ_2 indicate that the differentiation is to be applied only to ϕ_1 or ϕ_2.

By plugging (28) into (27) we finally obtain the desired asymptotic power series in $\hbar$ of the symbol $\Theta_N(t)$ of the product formula $\tilde{V}_N(t)$ in (25):

$$\Theta_N(x,\xi;t) \sim \sum_{j=0}^{+\infty} \hbar^j\,\Theta_{j,N}(x,\xi;t), \qquad (30)$$

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where

$$\Theta_{j,N} := \frac{\mathrm{i}^j}{2^j j!} \sum_{j_1,\ldots,j_N} \delta_{j_1+\cdots+j_N,j}\, \vartheta_N \sharp_{j_N} (\vartheta_{N-1} \sharp_{j_{N-1}} (\cdots (\vartheta_1 \sharp_{j_1} \vartheta_0) \cdots)), \tag{31}$$

with $\delta_{k,l}$ the Kronecker delta. This is statement (ii) of theorem 2.

Observe that $\Theta_{j,N}$ is a function of x and ξ and t, namely, is a function of the initial position and momentum of the particle and of the total time of the experiment. We want to prove that at each order j, whatever the initial nonzero momentum, after a certain time the particle is no longer confined in the region of observation. Precisely, we will prove the last statement of theorem 2:

Proposition 1. *For every $\xi \in \mathbb{R}^n$, $\xi \neq 0$, one gets that, for all $t > T_\xi := m\delta(\Omega)/|\xi|$,*

$$\lim_{N\to+\infty} \Theta_{j,N}(x,\xi;t) = 0,$$

uniformly in $x \in \mathbb{R}^n$ and $j \in \mathbb{N}$.

Proof. Let us fix the initial momentum of the particle $\xi \neq 0$.

Consider first the case $j = 0$,

$$\Theta_{0,N} = \vartheta_N \cdots \vartheta_1 \vartheta_0. \tag{32}$$

By making use of (26) and (20), we get

$$\vartheta_k(x,\xi) = \chi_\Omega^{(N)} \left(x + \frac{kt\xi}{Nm} \right), \tag{33}$$

so that

$$\Theta_{0,N}(x,\xi;t) = \chi_\Omega^{(N)}(x + \xi t/m) \cdots \chi_\Omega^{(N)}(x + \xi t/(mN))\chi_\Omega^{(N)}(x). \tag{34}$$

Since by equation (21) $\chi_\Omega^{(N)}$ is a mollification of the characteristic function χ_Ω, we get that the supports satisfy the equation

$$\mathrm{supp}[\Theta_{0,N}(\cdot,\xi;t)] \subset \mathrm{supp}[\vartheta_N(\cdot,\xi)\,\vartheta(\cdot,\xi)] = \mathrm{supp}\big[\chi_\Omega^{(N)}(\cdot + \xi t/m)\,\chi_\Omega^{(N)}\big]. \tag{35}$$

Observe that

$$\mathrm{supp}\big[\chi_\Omega^{(N)}(\cdot + \xi t/m)\big] = \mathrm{supp}\big[\chi_\Omega^{(N)}\big] - \xi t/m \subset \Omega_{\varepsilon_N} - \xi t/m \tag{36}$$

by equation (21). Therefore, the support

$$\mathrm{supp}[\Theta_{0,N}(\cdot,\xi;t)] \subset (\Omega_{\varepsilon_N} - \xi t/m) \tag{37}$$

is empty if $t > T_\xi^N := m\delta(\Omega_{\varepsilon_N})/|\xi|$. See figure 2. Therefore, since $T_\xi^N \to T_\xi := m\delta(\Omega)/|\xi|$, we have proved that for any $t > T_\xi$

$$\Theta_{0,N}(\cdot,\xi;t) \equiv 0, \tag{38}$$

for sufficiently large N.

Let us consider now $j > 0$. Observe that

$$\mathrm{supp}[\Theta_{j,N}(\cdot,\xi;t)] \subset \mathrm{supp}[\Theta_{0,N}(\cdot,\xi;t)]; \tag{39}$$

therefore, also in this case we have that for $t > T_\xi$

$$\Theta_{j,N}(\cdot,\xi;t) \equiv 0, \tag{40}$$

for sufficiently large N.   □

Note that this result holds for all $N \in \mathbb{N}$ and $t \in \mathbb{R}$, thus in particular it holds if we restrict N and t as in the hypothesis of theorem 2.
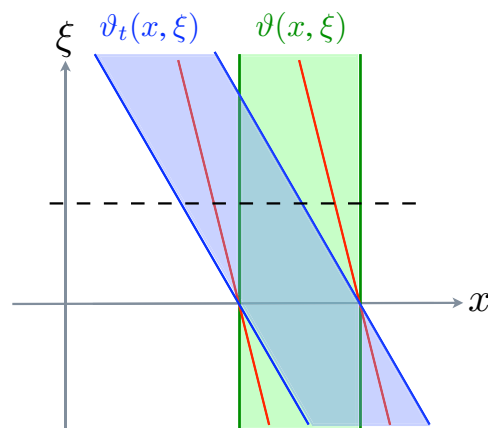
Figure 2. Phase space representation of $\Theta_{0,N}(x,\xi)$. The dashed line denotes the initial momentum $\xi \neq 0$ of the particle.

5. Concluding remarks

We have shown that the quantum Zeno effect vanishes at all orders in $\hbar$, when $\hbar \to 0$, and thus it is a purely quantum phenomenon without classical analog. Remark that, typically, quantum observables have instead non-zero asymptotic expansions in $\hbar$: elementary examples are (see e.g. [9], sections 50 and 51) the transition probabilities and the Bohr frequency condition. In the first case, the asymptotic expansion yields the quantum corrections to the classical observable evolved along the classical motion, and in the second case all quantum corrections to the classical frequencies. The quantum Zeno effect is at variance with the above examples. As such, it represents the counterpart of quantum tunneling through a confining barrier: in the quantum realm the first yields perfect localization, while the latter yields leakage and also the tunnelling amplitude vanishes to all orders in $\hbar$, and conversely in the classical realm. However, the analogy we have drawn is not yet totally symmetric. Indeed, quantum tunneling is known to be of order $e^{-1/\hbar}$. In this respect it would be very interesting to know whether the quantum Zeno effect is also exponentially vanishing.

Acknowledgment

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